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Approximation by an iterative method for regular solutions for incompressible fluids with mass diffusion

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Abstract

We study the approximation by means of an iterative method towards strong (and more regular) solutions for incompressible Navier–Stokes equations with mass diffusion. In addition, some convergence rates for the error between the approximation and the exact solution will be given, for weak, strong and more regular norms.

Keywords: Fluids with mass diffusion; Strong solutions; Iterative method; Convergence rates

1. Introduction

We use an iterative process in order to approximate solutions for a nonhomogeneous Navier–Stokes model with mass diffusion. The argument is:

(a) to obtain a priori estimations for the scheme sequence $(\rho^n, \mathbf{u}^n, p^n)$ (independent on n),
(b) to show that $(\rho^n, \mathbf{u}^n, p^n)$ is a Cauchy-sequence in an appropriate Banach space, and

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(c) to pass to the limit, proving that the limit $(\rho, \mathbf{u}, p)$ is the solution of the problem and obtaining some convergence rates.

1.1. The model

We consider the motion of a viscous fluid consisting in two components, for instance, saturated salt water and water. Some physical discussions and derivation of equations can be seen in Frank and Kamenetskii [3], Kazhikhov and Smagulov [7], Antoncev, Kazhikhov and Monakhov [1]. Let us give here a brief sketch.

Let the motion takes place in $\Omega \subset \mathbb{R}^3$ a bounded regular domain, and in a time interval $[0, T]$. Let ρ_1 and ρ_2 be the two characteristics densities (constants) of the two components, $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ their velocities and $e(t, \mathbf{x}), d(t, \mathbf{x})$ the mass and volume concentration of the first fluid ($1-e$, $1-d$ for the second one). Then, if we define the mean density $\rho(t, \mathbf{x}) = d\rho_1 + (1-d)\rho_2$, and the mean-volume and mean-mass velocities $\mathbf{u} = d\mathbf{v}^{(1)} + (1-d)\mathbf{v}^{(2)}$, $\mathbf{w} = e\mathbf{v}^{(1)} + (1-e)\mathbf{v}^{(2)}$, then the equations of motion in $Q_T = \Omega \times (0, T)$ are given by

$$\begin{cases} \rho(\mathbf{w}_t + \mathbf{w}\cdot\nabla\mathbf{w}) - \mu\Delta\mathbf{w} - (\mu+\mu')\nabla \operatorname{div}\mathbf{w} + \nabla P = \rho\mathbf{f} & \text{in } Q_T, \\ \operatorname{div}\mathbf{u} = 0, \quad \rho_t + \operatorname{div}(\rho\mathbf{w}) = 0 & \text{in } Q_T, \end{cases}$$

where P is the pressure and μ, μ' are viscosity constants such that $\mu > 0$ and $3\mu' + 2\mu > 0$. Here, $\mathbf{w}_t$ denotes the time derivative of $\mathbf{w}$, ∇ and Δ are the $3D$ gradient and Laplacian operators. Finally, div is the divergence operator.

On the other hand, Fick's diffusion law (see [3]) gives $\mathbf{w} = \mathbf{u} - \lambda\rho^{-1}\nabla\rho$, being $\lambda > 0$ the mass diffusion coefficient. Eliminating $\mathbf{w}$ in the preceding equations (see [7]), one arrives at the problem: To find $(\rho, \mathbf{u}, p)$ such that

$$\begin{cases} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \mu\Delta\mathbf{u} - \lambda\big((\mathbf{u}\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)\mathbf{u}\big) + \nabla p \\ \quad -\lambda^2\dfrac{1}{\rho}\left((\nabla\rho\cdot\nabla)\nabla\rho - \dfrac{1}{\rho}|\nabla\rho|^2\nabla\rho + \nabla\rho\Delta\rho\right) = \rho\mathbf{f} \text{ in } Q_T, \\ \operatorname{div}\mathbf{u} = 0 \text{ in } Q_T, \quad \mathbf{u}|_{\Sigma_T} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \\ \rho_t - \lambda\Delta\rho + \mathbf{u}\cdot\nabla\rho = 0 \text{ in } Q_T, \quad \left.\dfrac{\partial\rho}{\partial\mathbf{n}}\right|_{\Sigma_T} = 0, \quad \rho(0) = \rho_0 \text{ in } \Omega, \end{cases} \tag{1}$$

where $\Sigma_T = \partial\Omega \times (0, T)$. Here p is a potential function ($p = P + \lambda\mathbf{u}\cdot\nabla\rho - \lambda^2\Delta\rho + \lambda(2\mu + \mu')\Delta\log\rho$). Data of problem (1) are: initial data $(\rho_0, \mathbf{u}_0)$, external forces $\mathbf{f}$, viscosity and mass diffusion coefficients $\mu, \lambda > 0$.

Taking into account the equalities

$$(\mathbf{u}\cdot\nabla)\nabla\rho = \mathbf{u}_j\partial_j\partial_i\rho = \partial_i(\mathbf{u}_j\partial_j\rho) - \partial_i\mathbf{u}_j\partial_j\rho = \nabla(\mathbf{u}\cdot\nabla\rho) - \nabla\mathbf{u}\nabla\rho,$$

$$(\nabla\rho\cdot\nabla)\mathbf{u} = \partial_j\rho\partial_j\mathbf{u}_i = (\nabla\mathbf{u})^t\nabla\rho$$

(where $(\nabla\mathbf{u})^t$ is the transposed matrix of $\nabla\mathbf{u}$) and

$$\operatorname{div}\left(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho\right) = \frac{1}{\rho}\left((\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}|\nabla\rho|^2\nabla\rho + \nabla\rho\Delta\rho\right)$$

(where $\otimes$ denotes the tensorial product), the problem (1) admits the following re-formulation (with a new potential function $q = p - \lambda\mathbf{u}\cdot\nabla\rho$):

$$\begin{cases} \rho\big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) - \mu\Delta\mathbf{u} + \nabla q \\ \quad -\lambda\big((\nabla\mathbf{u})^t - \nabla\mathbf{u}\big)\nabla\rho - \lambda^2 \operatorname{div}\left(\dfrac{1}{\rho}\nabla\rho\otimes\nabla\rho\right) = \rho\mathbf{f} \text{ in } Q_T, \\ \operatorname{div}\mathbf{u} = 0 \text{ in } Q_T, \quad \mathbf{u}|_{\Sigma_T} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \\ \rho_t - \lambda\Delta\rho + \mathbf{u}\cdot\nabla\rho = 0 \text{ in } Q_T, \quad \left.\dfrac{\partial\rho}{\partial\mathbf{n}}\right|_{\Sigma_T} = 0, \quad \rho(0) = \rho_0 \text{ in } \Omega. \end{cases} \tag{2}$$

In this paper, we will always assume the hypothesis: there exist some constants $m, M > 0$, such that

$$0 < m \leqslant \rho_0 \leqslant M \quad \text{in } \Omega. \tag{3}$$

An interesting open problem is to extend the results of this paper to the case $m = 0$, i.e., assuming only $0 \leqslant \rho_0 \leqslant M$ in Ω.

1.2. *Known results*

Concerning a reduced model in $\Omega \subset \mathbb{R}^3$ (where the λ^2-terms of (1) are vanished), Kazhikhov and Smagulov [7] prove, using a semi-Galerkin method, the global existence of weak solutions and local strong solutions under hypothesis (3) and the following assumption about the viscosity and diffusion coefficients: $\lambda < 2\mu/(M-m)$. Also via this method, Salvi [9] proves the global (in time) existence of weak solutions in cylindrical and noncylindrical domains in $\mathbb{R}^n$ (n arbitrary) and with $m = 0$ in (3). On the other hand, Secchi in [12] studies the case $\Omega = \mathbb{R}^3$, proving local existence and uniqueness of strong solutions, using a fixed point argument.

For the full model (1) considered in this paper (including λ^2 terms), Beirão da Veiga [2] and Secchi [11], established the local existence of strong solutions by using linearization and fixed point argument. Indeed, in [2] Beirão da Veiga prove the global existence for a linearized version of the full model and using a fixed point argument the local existence of the nonlinear full model (1). No global results are available in general. In [11], λ/μ small enough is imposed, in order to show the existence and uniqueness of global solution in the 2-dimensional case. Moreover, it is showed the convergence, as $\lambda \to 0$, towards a weak solution of the Navier–Stokes problem with variable density. In the 3-dimensional case, global existence and convergence (as $\lambda \to 0$) towards Navier–Stokes with variable density is proven in [5], imposing only positive initial density ($\rho_0 \geqslant 0$).

1.3. *Space functions and equivalent norms*

We introduce standard spaces of the Navier–Stokes framework:

$$H = \big\{\mathbf{u}\colon\ \mathbf{u} \in L^2(\Omega)^3,\ \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\big\},$$
$$V = \big\{\mathbf{u}\colon\ \mathbf{u} \in H^1(\Omega)^3,\ \operatorname{div}\mathbf{u} = 0,\ \mathbf{u} = 0 \text{ on } \partial\Omega\big\},$$
$$L^2_0(\Omega) = \left\{p\colon\ p \in L^2(\Omega),\ \int_\Omega p(\mathbf{x}) = 0\right\}.$$

The norms $\|\mathbf{u}\|_{H^1}$ and $\|\nabla\mathbf{u}\|_{L^2}$ are equivalent in V, and $\|\mathbf{u}\|_{H^2}$ and $\|\Delta\mathbf{u}\|_{L^2}$ are equivalent in $H^2(\Omega) \cap V$ [8,13]. On the other hand, the norms $\|p\|_{H^1}$ and $\|\nabla p\|_{L^2}$ are equivalent in $H^1(\Omega) \cap L^2_0(\Omega)$.

On the other hand, for the density, let us consider the affine space ($k=2,3$)

$$H^k_N(\Omega)=\left\{\rho\in H^k(\Omega)\colon\ \frac{\partial\rho}{\partial\mathbf{n}}=0 \text{ on } \partial\Omega,\ \int_\Omega \rho(\mathbf{x})=\int_\Omega \rho_0(\mathbf{x})\right\}.$$

Obviously, $H^k_N(\Omega)=\overline{\rho_0}+H^k_{N,0}(\Omega)$, where $\overline{\rho_0}=(1/|\Omega|)\int_\Omega \rho_0(\mathbf{x})\,d\mathbf{x}$ and

$$H^k_{N,0}(\Omega)=\left\{\rho\in H^k(\Omega)\colon\ \frac{\partial\rho}{\partial\mathbf{n}}=0 \text{ on } \partial\Omega,\ \int_\Omega \rho(\mathbf{x})=0\right\}.$$

Therefore, $H^k_{N,0}(\Omega)$ ($k=2$ or $k=3$) is a closed subspace of $H^k_N(\Omega)$. Consequently, thanks to the H^2 and H^3 regularity of the Poisson–Neumann problem, norms $\|\rho\|_{H^2}$ and $\|\Delta\rho\|_{L^2}$ are equivalent in $H^2_N(\Omega)$ and $\|\rho\|_{H^3}$ and $\|\nabla\Delta\rho\|_{L^2}$ are equivalent in $H^3_N(\Omega)$ [2].

1.4. Exact solution and the iterative scheme

Assuming $\mathbf{u}_0\in V$, $\rho_0\in H^2_N(\Omega)$ satisfying (3) and $\mathbf{f}\in L^2(0,T;L^2(\Omega)^3)$, we are going to consider the (unique) strong solution $(\rho,\mathbf{u},p)$ of (1) defined in some (maybe small) time interval $(0,T)$ [1,2]: that is, $\rho\in L^2(0,T;H^3_N(\Omega))\cap C([0,T];H^2_N(\Omega))$, $\rho_t\in L^2(0,T;H^1(\Omega))$, $\mathbf{u}\in L^2(0,T;H^2(\Omega)^3)\cap C([0,T];V)$, $\mathbf{u}_t\in L^2(0,T;H)$, $p\in L^2(0,T;H^1(\Omega)\cap L^2_0(\Omega))$, verifying PDE equations a.e. in Q_T, boundary and initial conditions for $\rho,\mathbf{u}$ in the sense of spaces $H^2_N(\Omega)$ and V, respectively. It is easy to deduce that $(\rho,\mathbf{u},p)$ is the strong solution of (1) if and only if $(\rho,\mathbf{u},q)$ is the strong solution of (2).

Now, we introduce the iterative scheme that we will consider in this work, which solution $(\rho^n,\mathbf{u}^n,q^n)$ will be convergent towards the strong solution $(\rho,\mathbf{u},q)$ of (2):

Initialization: Let $\mathbf{u}^0(t)=\mathbf{u}_0$ for each $t\in[0,T]$.
Step $n\geqslant 1$: First, given $\mathbf{u}^{n-1}$, to find ρ^n such that

$$\rho^n_t+\mathbf{u}^{n-1}\cdot\nabla\rho^n-\lambda\Delta\rho^n=0,\quad \rho^n\big|_{t=0}=\rho_0\quad\text{and}\quad \left.\frac{\partial\rho^n}{\partial\mathbf{n}}\right|_{\Sigma_T}=0. \tag{4}$$

Afterwards, given $\mathbf{u}^{n-1}$ and ρ^n, to find $(\mathbf{u}^n,q^n)$ such that

$$\begin{cases} \rho^n\mathbf{u}^n_t+\left(\rho^n\mathbf{u}^{n-1}\cdot\nabla\right)\mathbf{u}^n-\mu\Delta\mathbf{u}^n+\nabla q^n-\lambda\left(\left(\nabla\mathbf{u}^n\right)^t-\nabla\mathbf{u}^n\right)\nabla\rho^n\\ \quad=\lambda^2\operatorname{div}\left(\dfrac{1}{\rho^n}\nabla\rho^n\otimes\nabla\rho^n\right)+\rho^n\mathbf{f},\\ \operatorname{div}\mathbf{u}^n=0,\quad \mathbf{u}^n\big|_{\Sigma_T}=0,\quad \mathbf{u}^n\big|_{t=0}=\mathbf{u}_0. \end{cases} \tag{5}$$

With this iterative scheme, we have reduced the nonlinear coupled system (2) into a sequence of linear decoupled problems (4) and (5). Existence, regularity and uniqueness of ρ^n solution of (4) and $(\mathbf{u}^n,q^n)$ solution of (5), can be easily obtained. For instance, in problem (5) one can made an argument similar to that given in the proof of Theorem 2.1 in [2], for an evolutive Stokes system modified by ρ^n in the time derivative term. Another possibility (see for instance [10]) is to use a Galerkin method, obtaining a sequence of finite-dimensional in space problems which can be rewritten as a Cauchy problem for an ordinary differential system. Then, local in time existence of Galerkin solutions is obtained from standard theory of Cauchy problems, and these solutions can be prolonged globally in time thanks to some a priori estimates. Finally, by a limit process, existence of global in time solution of (5) is deduced.

1.5. Main results of this paper

We will denote by (f, g) the inner product in $L^2(\Omega)$, by $|f|_2$ the $L^2(\Omega)$-norm and by $|f|_p$ the $L^p(\Omega)$-norm ($1 \leqslant p \leqslant +\infty$). Moreover, $\|f\|_k$ will denote the $H^k(\Omega)$-norm ($k \geqslant 0$). In particular, $|\cdot|_2 = \|\cdot\|_0$. Any other norm in a space $X(\Omega)$ defined in Ω will be denoted by $\|f\|_X$. Finally, for a Cartesian product $X \times Y$, we will denote $\|(x, y)\|_{X\times Y} = \max\{\|x\|_X, \|y\|_Y\}$.

Our goal in this paper is double: to prove that $(\rho^n, \mathbf{u}^n, q^n)$ is a Cauchy sequence in a suitable Banach space which converges towards the strong solution $(\rho, \mathbf{u}, q)$ of problem (2), and to give some estimates of the convergence rates.

More precisely, we will prove the following four main results, corresponding with the convergence rates with respect to the weak norms, strong norms and more regular norms, see (19) for definition of bound $G(n)$. In all the cases, the following "smallness conditions" on data must be imposed:

Considering $C_1, C_2, C_3, C_4 > 0$ the constants furnished in Lemma 3.1 (see below in Section 3), we assume that there exist $K_1, K_2 > 0$ such that

$$\lambda|\Delta\rho_0|_2^2 \exp\left(\frac{C_1}{\lambda^3} K_1^2 T\right) \leqslant K_2, \tag{6}$$

$$\left(\mu|\nabla\mathbf{u}_0|_2^2 + C_2 \int_0^T |\mathbf{f}|_2^2 + C_4\big(\lambda K_2^3 T + \lambda^{3/2} K_2^2 T^{1/2}\big)\right) \exp\big(C_3\big(K_1^2 + \lambda^2 K_2^2\big)T\big) \leqslant K_1. \tag{7}$$

Notice that hypotheses (6)–(7) are either smallness restrictions on the data $(\mathbf{f}, \mathbf{u}_0, \rho_0)$ (taking K_1 and K_2 small enough), or smallness conditions on the final time T (taking any $K_2 > \lambda|\Delta\rho_0|^2$ and $K_1 > \mu|\nabla\mathbf{u}_0|^2$). For the simplify model without λ^2 terms ($C_4 = 0$), it is easy to verify that (6)–(7) do not imply smallness constraints on ρ_0.

Theorem 1.1. *Under constraints* (6)–(7) *and regularity hypotheses on data of Theorem* 3.2 (*see Section* 3)*, one has existence* (*and uniqueness*) *of the strong solution* $(\rho, \mathbf{u})$ *of problem* (2)*, which is obtained as the limit of the sequence* $(\rho^n, \mathbf{u}^n)$*. Moreover, the following error estimates* (*in weak norms*) *hold for all* $t \in [0, T]$:

$$\big\|\big(\rho^n - \rho, \mathbf{u}^n - \mathbf{u}\big)(t)\big\|_{H^1\times L^2}^2 \leqslant G(n), \tag{8}$$

$$\int_0^t \Big(\big\|\big(\rho^n - \rho, \mathbf{u}^n - \mathbf{u}\big)(\tau)\big\|_{H^2\times H^1}^2 + \big\|\big(\rho_t^n - \rho_t\big)(\tau)\big\|_{L^2}^2\Big)\, d\tau \leqslant G(n). \tag{9}$$

Theorem 1.2. *Under hypotheses of Theorem* 1.1*, the following error estimates* (*in strong norms*) *hold for all* $t \in [0, T]$:

$$\big\|\big(\rho^n - \rho, \mathbf{u}^n - \mathbf{u}\big)(t)\big\|_{H^2\times H^1}^2 + \int_0^t \big\|\big(\rho_t^n - \rho_t, \mathbf{u}_t^n - \mathbf{u}_t\big)\big\|_{H^1\times L^2}^2 \leqslant G(n), \tag{10}$$

$$\int_0^t \big\|\big(\rho^n - \rho, \mathbf{u}^n - \mathbf{u}, q^n - q\big)\big\|_{H^3\times H^2\times H^1}^2 \leqslant G(n). \tag{11}$$

Theorem 1.3. *Under hypotheses of Theorem* 3.3 (*see Section* 3), *one has that the strong solution* $(\rho, \mathbf{u}, q)$ *given in Theorem* 1.2 *is more regular, concretely*

$$(\rho, \mathbf{u}, q) \in L^\infty\big(H^3 \times H^2 \times H^1\big) \cap L^2\big(H^4 \times H^3 \times H^2\big),$$
$$(\rho_t, \mathbf{u}_t) \in L^\infty\big(H^1 \times L^2\big) \cap L^2\big(H^2 \times H^1\big), \quad \rho_{tt} \in L^2\big(L^2\big).$$

Moreover, the following error estimates for density hold for all $t \in [0, T]$:

$$\big\|\big(\rho_t^n - \rho_t\big)(t)\big\|_{H^1}^2 + \int_0^t \big\|\big(\rho_t^n - \rho_t\big)(\tau)\big\|_{H^2}^2\, d\tau \leqslant G(n-1), \tag{12}$$

$$\big\|\big(\rho^n - \rho\big)(t)\big\|_{H^3}^2 + \int_0^t \big\|\big(\rho^n - \rho\big)(\tau)\big\|_{H^4}^2\, d\tau \leqslant G(n-1). \tag{13}$$

Theorem 1.4. *Under hypotheses of Theorem* 3.4 (*see Section* 3), *one has that the solution* $(\rho, \mathbf{u}, q)$ *given in Theorem* 1.3 *is more regular, concretely*

$$\big(\rho, \sqrt{\sigma(t)}\mathbf{u}, \sqrt{\sigma(t)}q\big) \in L^\infty\big(H^4 \times H^3 \times H^2\big) \cap L^2\big(H^5 \times H^4 \times H^3\big),$$
$$\big(\rho_t, \sqrt{\sigma(t)}\mathbf{u}_t\big) \in L^\infty\big(H^2 \times H^1\big) \cap L^2\big(H^3 \times H^2\big),$$
$$\big(\rho_{tt}, \sqrt{\sigma(t)}\mathbf{u}_{tt}\big) \in L^2\big(H^1 \times L^2\big),$$

where $\sigma(t) = \min\{t, 1\}$ (*the regularity for velocity and pressure will be valid only for strictly positive times*). *Moreover, the following error estimates hold for all* $t \in [0, T]$:

$$\sigma(t)\big\|\big(\mathbf{u}_t^n - \mathbf{u}_t\big)(t)\big\|_{L^2}^2 + \int_0^t \sigma(\tau)\big\|\big(\mathbf{u}_t^n - \mathbf{u}_t\big)(\tau)\big\|_{H^1}^2\, d\tau \leqslant G(n-1), \tag{14}$$

$$\sigma(t)\big\|\big(\mathbf{u}^n - \mathbf{u}, q^n - q\big)(t)\big\|_{H^2 \times H^1}^2 \leqslant G(n-1), \tag{15}$$

$$\int_0^t \sigma(\tau)\big\|\big(\mathbf{u}^n - \mathbf{u}, q^n - q\big)(\tau)\big\|_{H^3 \times H^2}^2\, d\tau \leqslant G(n-1). \tag{16}$$

Notice that convergence rates in weak norms given in Theorem 1.1 are the same as those in strong norms given in Theorem 1.2 (even under the same hypotheses). But, convergence rates for regular norms given in Theorems 1.3 and 1.4 change from $G(n)$ to $G(n-1)$ (and more hypotheses on data are necessary).

2. Some estimates of Gronwall's type

The following well known Gronwall's lemma will be frequently used:

Lemma 2.1 *(Gronwall). Let* a, b, c, d *be positive* $L^1(0, T)$ *functions satisfying the differential inequality*: $a'(t) + b(t) \leqslant c(t)a(t) + d(t)$ *a.e.* $t \in (0, T)$. *Then, for any* $t \in (0, T)$:

$$a(t) + \int_0^t b(s)\, ds \leqslant \left(a(0) + \int_0^t d(s)\, ds\right) \exp\left(\int_0^t c(s)\, ds\right).$$

Now, we present a more specific estimate of Gronwall's type, which will be used in the sequel, in order to obtain either scheme estimates or error estimates.

Lemma 2.2 *(Gronwall with recurrence). Let* $(a_n), (b_n)$ *be two sequences of positive* $L^1(0,T)$ *functions such that* $a_n(0) \leqslant A \in \mathbb{R}$ *and satisfying*

$$a_n'(t) + b_n(t) \leqslant c_n(t)a_n(t) + d_n(t)a_{n-1}(t) \quad \text{a.e. } t \in (0,T), \tag{17}$$

where $(c_n), (d_n)$ *are two sequences of positive functions, bounded in* $L^1(0,T)$ *and* $L^2(0,T)$, *respectively. Then, there exist two constants* $D > 0$ *and* $E > 0$ *independent on* n (*depending on bounds of* $\|c_n\|_{L^1(0,T)}$ *and* $\|d_n\|_{L^2(0,T)}$) *such that for any* $t \in (0,T)$ *and for any* $n \geqslant 1$, *one has*:

$$a_n(t) + \int_0^t b_n(s)\,ds \leqslant E\left(A\,\mathrm{e}^{Dt/2} + \|a_0\|_{L^\infty(0,t)}\left[\frac{(Dt)^n}{n!}\right]^{1/2}\right).$$

Proof. Applying Gronwall's lemma to (17) (recalling that $a_n(0) \leqslant A$) one has the estimate:

$$\begin{aligned} a_n(t) + \int_0^t b_n(s)\,ds &\leqslant \left(A + \int_0^t d_n(s)a_{n-1}(s)\,ds\right)\exp\left(\int_0^t c_n\right) \\ &\leqslant C\left(A + \left[\int_0^t \left|a_{n-1}(s)\right|^2 ds\right]^{1/2}\right). \end{aligned} \tag{18}$$

Therefore, if we define $\tilde{a}_n(t) = |a_n(t)|^2$, one has

$$\tilde{a}_n(t) \leqslant D\left(A^2 + \int_0^t \tilde{a}_{n-1}(s)\,ds\right)$$

hence, by means of an induction argument (applying Fubini's theorem),

$$\begin{aligned} \tilde{a}_n(t) &\leqslant DA^2\left(1 + Dt + \cdots + \frac{(D\,t)^{n-1}}{(n-1)!}\right) + D^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\tilde{a}_0(s)\,ds \\ &\leqslant DA^2\mathrm{e}^{Dt} + D^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}a_0^2(s)\,ds \leqslant DA^2\mathrm{e}^{Dt} + \|a_0\|^2_{L^\infty(0,t)}\frac{(Dt)^n}{n!}. \end{aligned}$$

Finally, returning to (18) and applying previous estimates for $a_{n-1}^2(s)$, one has

$$\int_0^t b_n(s)\,ds \leqslant C\left(A\left(1 + \mathrm{e}^{Dt/2}\right) + \|a_0\|_{L^\infty(0,t)}\left[\int_0^t \frac{(Ds)^{n-1}}{(n-1)!}\right]^{1/2}\right)$$

hence we can finish the proof of this lemma. □

Remark 2.3. In this paper, we will use the previous lemma in two situations, in order to obtain either scheme estimates, using in particular that $a_n(t)+\int_0^t b_n(s)\,ds \leqslant C$, or error estimates where $A=0$ and then, for each $n \geqslant 1$,

$$a_n(t)+\int_0^t b_n(s)\,ds \leqslant G(n) := C\big[(Dt)^n/n!\big]^{1/2}. \tag{19}$$

Here and in the sequel, we will denote by C different constants, always independent on n.

3. Scheme estimates

In this section, the task is to prove some estimates (uniformly respect to n) for the sequence $(\rho^n, \mathbf{u}^n, p^n)$.

The following classical "interpolation and Sobolev" inequality will be used:

$$|f|_3 \leqslant |f|_2^{1/2}|f|_6^{1/2} \leqslant C\|f\|_0^{1/2}\|f\|_1^{1/2}.$$

In particular, $|f\cdot g|_2 \leqslant |f|_3|g|_6 \leqslant C\|f\|_0^{1/2}\|f\|_1^{1/2}\|g\|_1$. Moreover, we will use the following more specific interpolation inequality [4]:

$$|f|_\infty \leqslant C\|f\|_1^{1/2}\|f\|_2^{1/2}.$$

In particular, $|f\cdot g|_2 \leqslant |f|_\infty|g|_2 \leqslant C\|f\|_1^{1/2}\|f\|_2^{1/2}\|g\|_0$. From previous inequalities, one has

$$\big|\nabla(fg)\big|_2 \leqslant \big|(\nabla f)g\big|_2 + |f\nabla g|_2 \leqslant C\|f\|_1^{1/2}\|f\|_2^{1/2}\|g\|_1. \tag{20}$$

The "maximum principle" for the ρ^n-problem (4) jointly with the hypothesis (3) imply [2]

$$0<m\leqslant \rho^n(x,t)\leqslant M \quad \text{in } Q_T. \tag{21}$$

Lemma 3.1. *There exist some positive constants $\beta, C_1, C_2, C_3, C_4$ (depending on m, M, μ, Ω but independent on n and λ) such that, for any $n \geqslant 1$,*

$$\lambda\frac{d}{dt}\big|\Delta\rho^n\big|_2^2 + \frac{\lambda^2}{4}\big|\nabla\Delta\rho^n\big|_2^2 + \frac{1}{2}\big|\nabla\rho_t^n\big|_2^2 \leqslant \frac{C_1}{\lambda^3}\big(\mu\big|\nabla\mathbf{u}^{n-1}\big|_2^2\big)^2\lambda\big|\Delta\rho^n\big|_2^2, \tag{22}$$

$$\begin{aligned}
&\mu\frac{d}{dt}\big|\nabla\mathbf{u}^n\big|_2^2 + \frac{m}{2}\big|\mathbf{u}_t^n\big|_2^2 + \beta\big(\big|\Delta\mathbf{u}^n\big|_2^2 + \big|\nabla q^n\big|_2^2\big) \leqslant C_2|\mathbf{f}|_2^2 \\
&\quad + C_3\big(\big(\mu\big|\nabla\mathbf{u}^{n-1}\big|_2^2\big)^2 + \lambda^2\big(\lambda\big|\Delta\rho^n\big|_2^2\big)^2\big)\mu\big|\nabla\mathbf{u}^n\big|_2^2 \\
&\quad + C_4\big(\lambda\big(\lambda\big|\Delta\rho^n\big|_2^2\big)^3 + \lambda^{3/2}\big(\lambda\big|\Delta\rho^n\big|_2^2\big)^{3/2}\lambda\big|\nabla\Delta\rho^n\big|_2\big).
\end{aligned} \tag{23}$$

Proof. Multiplying the density equation (4) by $-\Delta\rho_t^n$, and taking gradient of (4) multiplied by $-\lambda\nabla\Delta\rho^n$, integrating by parts in Ω (all boundary terms vanish, thanks to the Neumann boundary condition for ρ^n) and using (20),

$$\begin{aligned}
&\lambda\frac{d}{dt}\big|\Delta\rho^n\big|_2^2 + \frac{\lambda^2}{2}\big|\nabla\Delta\rho^n\big|_2^2 + \frac{1}{2}\big|\nabla\rho_t^n\big|_2^2 \\
&\quad\leqslant C\big|\nabla\big(\mathbf{u}^{n-1}\cdot\nabla\rho^n\big)\big|_2^2 \\
&\quad\leqslant C\big\|\mathbf{u}^{n-1}\big\|_1^2\big\|\rho^n\big\|_2\big\|\rho^n\big\|_3 \leqslant \varepsilon\lambda^2\big\|\rho^n\big\|_3^2 + \frac{C_\varepsilon}{\lambda^2}\big\|\mathbf{u}^{n-1}\big\|_1^4\big\|\rho^n\big\|_2^2.
\end{aligned}$$

Then, recalling equivalent norms $|\Delta\rho^n|_2 \sim \|\rho^n\|_2$ and $|\nabla\Delta\rho^n|_2 \sim \|\rho^n\|_3$, the first inequality (22) of this lemma holds.

To prove the second inequality (23), one rewrites (5) as the following evolutionary Stokes problem

$$\rho\mathbf{u}_t^n - \mu\Delta\mathbf{u}^n - \nabla q^n = \mathbf{F}, \qquad \operatorname{div}\mathbf{u}^n = 0, \qquad \mathbf{u}^n|_\Sigma = 0, \qquad \mathbf{u}^n(0) = \mathbf{u}_0, \tag{24}$$

where

$$\mathbf{F} = -\big(\rho^n\mathbf{u}^{n-1}\cdot\nabla\big)\mathbf{u}^n + \lambda\big(\big(\nabla\mathbf{u}^n\big)^t - \nabla\mathbf{u}^n\big)\nabla\rho^n + \lambda^2\operatorname{div}\left(\frac{1}{\rho^n}\nabla\rho^n\otimes\nabla\rho^n\right) + \rho^n\mathbf{f}.$$

Taking $\mathbf{u}_t^n$ as tests function in (24),

$$\mu\frac{d}{dt}\big|\nabla\mathbf{u}^n\big|_2^2 + m\big|\mathbf{u}_t^n\big|_2^2 \leqslant C|\mathbf{F}|_2^2. \tag{25}$$

We bound $|\mathbf{F}|_2^2$ using inequality (20) and some equivalent norms:

$$\begin{aligned}
|\mathbf{F}|_2^2 &\leqslant C\big(|\mathbf{f}|_2^2 + \big|\big(\mathbf{u}^{n-1}\cdot\nabla\big)\mathbf{u}^n\big|_2^2 + \lambda^2\big|\big(\big(\nabla\mathbf{u}^n\big)^t - \nabla\mathbf{u}^n\big)\nabla\rho^n\big|_2^2\big) \\
&\quad + C\lambda^4\left|\nabla\cdot\left(\frac{1}{\rho^n}\nabla\rho^n\otimes\nabla\rho^n\right)\right|_2^2 \\
&\leqslant C\big(|\mathbf{f}|_2^2 + \big\|\mathbf{u}^{n-1}\big\|_1^2\big\|\mathbf{u}^n\big\|_1\big\|\mathbf{u}^n\big\|_2 + \lambda^2\big\|\nabla\rho^n\big\|_1^2\big\|\mathbf{u}^n\big\|_1\big\|\mathbf{u}^n\big\|_2\big) \\
&\quad + C\lambda^4\big(\big|\nabla\rho^n\big|_6^6 + \big|\nabla\rho^n\nabla\nabla\rho^n\big|_2^2\big) \\
&\leqslant C|\mathbf{f}|_2^2 + \varepsilon\big|\Delta\mathbf{u}^n\big|_2^2 + C_\varepsilon\big(\big|\nabla\mathbf{u}^{n-1}\big|_2^4 + \lambda^4\big|\Delta\rho^n\big|_2^4\big)\big|\nabla\mathbf{u}^n\big|_2^2 \\
&\quad + C\lambda^4\big(\big|\Delta\rho^n\big|_2^6 + \big|\Delta\rho^n\big|_2^3\big|\nabla\Delta\rho^n\big|_2\big).
\end{aligned}$$

In order to estimate the $H^2(\Omega)$-norm for the velocity $\mathbf{u}^n$ and the $H^1(\Omega)$-norm for the pressure q^n, we use that $(\mathbf{u}^n, q^n)$ is the solution of a stationary Stokes equations (considering in (24) the term $\rho\mathbf{u}_t^n$ on the right-hand side). Then, the classical $H^2\times H^1$ regularity results of the Stokes problem [8,13] and previous bounds for $|\mathbf{F}|_2^2$, yield:

$$\begin{aligned}
\big|\Delta\mathbf{u}^n\big|_2^2 + \big|\nabla q^n\big|_2^2 &\leqslant C\big|\mathbf{u}_t^n\big|_2^2 + C|\mathbf{f}|_2^2 + \varepsilon\big|\Delta\mathbf{u}^n\big|_2^2 \\
&\quad + C_\varepsilon\big(\big|\nabla\mathbf{u}^{n-1}\big|_2^4 + \lambda^4\big|\Delta\rho^n\big|_2^4\big)\big|\nabla\mathbf{u}^n\big|_2^2 \\
&\quad + C\lambda^4\big(\big|\Delta\rho^n\big|_2^6 + \big|\Delta\rho^n\big|_2^3\big|\nabla\Delta\rho^n\big|_2\big).
\end{aligned}\tag{26}$$

Choosing ε small enough and making an appropriate "balance" between (25) and (26) in order to eliminate the term $|\mathbf{u}_t^n|_2^2$ at the right-hand side, one can arrive to the second inequality (23) of this lemma. □

As a consequence of the previous lemma, by means of a standard induction argument jointly with Gronwall's lemma, we arrive at the following.

Theorem 3.2. *Assume $\mathbf{u}_0 \in V$, $\rho_0 \in H_N^2(\Omega)$ satisfying* (3) *and* $\mathbf{f} \in L^2(Q_T)^3$, *such that the smallness hypotheses* (6)–(7) *hold, then, the following inequalities hold, for any $n \geqslant 1$ and for all $t \in (0, T)$,*

$$\lambda\left|\Delta\rho^n(t)\right|_2^2+\int_0^t\left(\frac{\lambda^2}{4}\left|\nabla\Delta\rho^n(\tau)\right|_2^2+\frac{1}{2}\left|\nabla\rho_t^n(\tau)\right|_2^2\right)d\tau\leqslant K_2, \tag{27}$$

$$\mu\left|\nabla\mathbf{u}^n(t)\right|_2^2+\int_0^t\left(\frac{m}{2}\left|\mathbf{u}_t^n(\tau)\right|_2^2+\beta\left(\left|\Delta\mathbf{u}^n(\tau)\right|_2^2+\left|\nabla q^n(\tau)\right|_2^2\right)\right)d\tau\leqslant K_1. \tag{28}$$

In particular, taking into account equivalent norms in V, $H^2\cap V$, H^2_N and H^3_N, it suffices to prove (27) *and* (28)*, the following estimates hold*:

$$\left(\rho^n,\mathbf{u}^n\right)\quad in\ L^\infty\left(H^2\times H^1\right)\cap L^2\left(H^3\times H^2\right), \tag{29}$$

$$\left(\rho_t^n,\mathbf{u}_t^n\right)\quad in\ L^2\left(H^1\times L^2\right),\qquad q^n\quad in\ L^2\left(H^1\right). \tag{30}$$

Now, we are going to obtain more regular scheme estimates. In fact, we will do weak and strong estimates of time derivatives functions $(\rho_t^n,\mathbf{u}_t^n,q_t^n)$. Differentiating (4) and (5) with respect to t, the problems satisfied by ρ_t^n and $(\mathbf{u}_t^n,q_t^n)$ are

$$\begin{cases}\eta_t+B(\bar{\mathbf{v}},\nabla\rho)+B(\bar{\mathbf{u}},\nabla\eta)=\lambda\Delta\eta,\quad \dfrac{\partial\eta}{\partial n}_{|\Sigma}=0,\quad \eta(0)=\rho_t^n(0),\\ \rho\left(\mathbf{v}_t+B(\bar{\mathbf{u}},\nabla\mathbf{v})+B(\bar{\mathbf{v}},\nabla\mathbf{u})\right)-\mu\Delta\mathbf{v}+\nabla q_t\\ \qquad -\lambda\left[C(\nabla\mathbf{u},\nabla\eta)+C(\nabla\mathbf{v},\nabla\rho)\right]\\ \qquad -\lambda^2\operatorname{div}\left[-\dfrac{1}{\rho^2}\eta\nabla\rho\otimes\nabla\rho+\dfrac{1}{\rho}(\nabla\eta\otimes\nabla\rho+\nabla\rho\otimes\nabla\eta)\right]\\ \quad =-\eta\mathbf{v}-\eta B(\bar{\mathbf{u}},\nabla\mathbf{u})+\eta\mathbf{f}+\rho\mathbf{f}_t,\\ \operatorname{div}\mathbf{v}=0,\quad \mathbf{v}|_\Sigma=0,\quad \mathbf{v}(0)=\mathbf{u}_t^n(0),\end{cases} \tag{31}$$

where we have denoted:

$$\mathbf{u}=\mathbf{u}^n,\qquad \bar{\mathbf{u}}=\mathbf{u}^{n-1},\qquad \rho=\rho^n,\qquad q=q^n,$$
$$\mathbf{v}=\mathbf{u}_t^n,\qquad \bar{\mathbf{v}}=\mathbf{u}_t^{n-1},\qquad \eta=\rho_t^n,$$

and $B(f,\nabla g)=(f\cdot\nabla)g$ and $C(\nabla f,\nabla g)=((\nabla f)^t-\nabla f)\nabla g$. Now, we will obtain scheme estimates with one order more of regularity than in Theorem 3.2, when data are more regular (but without additional restrictive hypothesis).

Theorem 3.3. *Assume hypotheses of Theorem* 3.2. *If $\mathbf{u}_0\in H^2(\Omega)\cap V$ and $\rho_0\in H^3_N(\Omega)$, then the following estimations hold*:

$$\rho_t^n\quad in\ L^\infty(H^1)\cap L^2\left(H^2\right),\qquad \rho_{tt}^n\quad in\ L^2\left(L^2\right), \tag{32}$$

$$\rho^n\quad in\ L^\infty\left(H^3\right)\cap L^2\left(H^4\right). \tag{33}$$

Moreover, if $\mathbf{f}\in L^2(0,T;L^2)$ with $\mathbf{f}(0)\in L^2(\Omega)$ and $\mathbf{f}_t\in L^2(0,T;L^{6/5})$, then:

$$\mathbf{u}_t^n\quad is\ bounded\ in\ L^\infty\left(L^2\right)\cap L^2\left(H^1\right). \tag{34}$$

In addition, if $\mathbf{f}\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$, one has

$$\left(\mathbf{u}^n,q^n\right)\quad is\ bounded\ in\ L^\infty\left(H^2\times H^1\right)\cap L^2\left(H^3\times H^2\right). \tag{35}$$

Proof. An outline of the proof is the following: (32) is obtained doing weak estimates of ρ_t^n problem and (33) is deduced from (32) and regularity results for the Poisson problem associated to ρ^n. Afterwards, (34) is obtained doing weak estimates of $(\mathbf{u}_t^n, q_t^n)$ problem and (35) is deduced from (34) and the Stokes problem associated to $(\mathbf{u}^n, q^n)$.

Multiplying Eq. $(31)_1$ respectively by η_t and $-\lambda\Delta\eta$, we get

$$\begin{aligned}\lambda\frac{d}{dt}\|\eta\|_1^2+\lambda^2\|\eta\|_2^2+\|\eta_t\|_0^2 &\leqslant C\big(\big|B(\bar{\mathbf{v}},\nabla\rho)\big|_2^2+\big|B(\bar{\mathbf{u}},\nabla\eta)\big|_2^2\big)\\ &\leqslant \varepsilon\|\eta\|_2^2+C_\varepsilon\|\bar{\mathbf{u}}\|_1^4\|\eta\|_1^2+\|\rho\|_2\|\rho\|_3\|\bar{\mathbf{v}}\|_0^2,\end{aligned}$$

where again (20) and Young's inequality have been applied. Taking into account estimates of Theorem 3.2, one has

$$\lambda\frac{d}{dt}\|\eta\|_1^2+\lambda^2\|\eta\|_2^2+\|\eta_t\|_0^2\leqslant C\big(\|\rho\|_3\|\bar{\mathbf{v}}\|_0^2+\|\eta\|_1^2\big). \tag{36}$$

For the regularity of ρ^n, we will use the Poisson problem

$$-\Delta\rho^n=-\rho_t^n-\mathbf{u}^{n-1}\cdot\nabla\rho^n\quad\text{in }\Omega,\qquad \frac{\partial\rho^n}{\partial n}\bigg|_{\partial\Omega}=0. \tag{37}$$

Using H^3-regularity of (37), we have

$$\begin{aligned}\|\rho^n\|_3 &\leqslant C\big(\|\rho_t^n\|_1+\|\mathbf{u}^{n-1}\cdot\nabla\rho^n\|_1\big)\leqslant C\big(\|\rho_t^n\|_1+\|\mathbf{u}^{n-1}\|_1\|\rho^n\|_2^{1/2}\|\rho^n\|_3^{1/2}\big)\\ &\leqslant \frac{1}{2}\|\rho^n\|_3+C\big(\|\rho_t^n\|_1+\|\mathbf{u}^{n-1}\|_1^2\|\rho^n\|_2\big)\end{aligned}$$

hence, using estimates of Theorem 3.2,

$$\|\rho^n\|_3\leqslant C\big(\|\rho_t^n\|_1+1\big),\quad \text{i.e., }\ \|\rho\|_3\leqslant C\big(\|\eta\|_1+1\big). \tag{38}$$

In particular, $\|\rho\|_3\|\bar{\mathbf{v}}\|_0^2\leqslant C(\|\bar{\mathbf{v}}\|_0^2+\|\bar{\mathbf{v}}\|_0^2\|\eta\|_1^2)$. Applying this inequality in (36), one has

$$\lambda\frac{d}{dt}\|\eta\|_1^2+\lambda^2\|\eta\|_2^2+\|\eta_t\|_0^2\leqslant C\big(\|\bar{\mathbf{v}}\|_0^2+\|\bar{\mathbf{v}}\|_0^2\|\eta\|_1^2+\|\eta\|_1^2\big). \tag{39}$$

In order to bound $\|\eta(0)\|_1^2$, we take H^1-norm in (4) evaluated at $t=0$:

$$\|\eta(0)\|_1^2=\|\rho_t^n(0)\|_1\leqslant C\big(\|\mathbf{u}_0\|_2\|\rho_0\|_3+\|\rho_0\|_3\big).$$

Therefore, hypotheses $\mathbf{u}_0\in H^2$ and $\rho_0\in H^3$ imply $\|\eta(0)\|_1^2\leqslant C$. Applying Gronwall's lemma to (39), since $\|\bar{\mathbf{v}}\|_0^2$ is bounded in $L^1(0,T)$, we deduce (32).

Using that (ρ_t^n) is bounded in $L^\infty(H^1)$ in (38), we get that ρ^n is bounded in $L^\infty(H^3)$. On the other hand, from H^4-regularity of (37),

$$\|\rho^n\|_4\leqslant C\big(\|\rho_t^n\|_2+\|\mathbf{u}^{n-1}\cdot\nabla\rho^n\|_2\big)\leqslant C\big(\|\rho_t^n\|_2+\|\mathbf{u}^{n-1}\|_2\|\rho^n\|_3\big).$$

Using the bounds $\mathbf{u}^{n-1}\in L^2(H^2)$, $\rho^n\in L^\infty(H^3)$ and $\rho_t^n\in L^2(H^2)$, we get that ρ^n is bounded in $L^2(H^4)$, hence (33) is completed.

To improve estimates for the velocity (and pressure), we multiply Eq. $(31)_2$ by $\mathbf{v}$, using the equality

$$\int_\Omega\rho\mathbf{v}_t\cdot\mathbf{v}+\int_\Omega\rho B(\bar{\mathbf{u}},\nabla\mathbf{v})\cdot\mathbf{v}-\lambda\int_\Omega(\nabla\rho\cdot\nabla)\mathbf{v}\cdot\mathbf{v}=\frac{1}{2}\frac{d}{dt}\int_\Omega\rho|\mathbf{v}|^2$$

(obtained thanks to (4) multiplied by $|\mathbf{v}|^2/2$), and we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{v}|^2 + \int_\Omega \rho B(\bar{\mathbf{v}}, \nabla\mathbf{u})\cdot\mathbf{v} + \mu\int_\Omega |\nabla\mathbf{v}|^2 + \int_\Omega \eta B(\bar{\mathbf{u}}, \nabla\mathbf{u})\cdot\mathbf{v} \\
&\quad - \lambda\int_\Omega C(\nabla\mathbf{u}, \nabla\eta)\cdot\mathbf{v} - \lambda\int_\Omega C(\nabla\mathbf{v}, \nabla\rho)\cdot\mathbf{v} + \lambda\int_\Omega (\nabla\rho\cdot\nabla)\mathbf{v}\cdot\mathbf{v} \\
&\quad + \lambda^2\int_\Omega \left[-\frac{1}{\rho^2}\eta\nabla\rho\otimes\nabla\rho + \frac{1}{\rho}(\nabla\eta\otimes\nabla\rho + \nabla\rho\otimes\nabla\eta)\right] : \nabla\mathbf{v} \\
&= -\int_\Omega \eta\mathbf{v}\cdot\mathbf{v} + \int_\Omega \eta f\cdot\mathbf{v} + \int_\Omega \rho f_t\cdot\mathbf{v}.
\end{aligned}$$

We estimate the previous terms:

$$\int_\Omega \rho B(\bar{\mathbf{v}}, \nabla\mathbf{u})\cdot\mathbf{v} \leqslant |\rho|_\infty|\bar{\mathbf{v}}|_2|\nabla\mathbf{u}|_3|\mathbf{v}|_6 \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\mathbf{u}\|_1\|\mathbf{u}\|_2\|\bar{\mathbf{v}}\|_0^2,$$

$$\int_\Omega \eta B(\bar{\mathbf{u}}, \nabla\mathbf{u})\cdot\mathbf{v} \leqslant |\eta|_6|\bar{\mathbf{u}}|_6|\nabla\mathbf{u}|_3|\mathbf{v}|_6 \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\bar{\mathbf{u}}\|_1^2\|\mathbf{u}\|_1\|\mathbf{u}\|_2\|\eta\|_1^2,$$

$$\lambda\int_\Omega C(\nabla\mathbf{u}, \nabla\eta)\cdot\mathbf{v} \leqslant |\nabla\mathbf{u}|_2|\nabla\eta|_3|\mathbf{v}|_6 \leqslant \varepsilon\big(\|\mathbf{v}\|_1^2 + \|\eta\|_2^2\big) + C_\varepsilon\|\mathbf{u}\|_1^4\|\eta\|_1^2,$$

$$\begin{aligned}
&\lambda\int_\Omega C(\nabla\mathbf{v}, \nabla\rho)\cdot\mathbf{v} - \lambda\int_\Omega (\nabla\rho\cdot\nabla)\mathbf{v}\cdot\mathbf{v} \\
&\quad \leqslant C|\nabla\rho|_\infty|\nabla\mathbf{v}|_2|\mathbf{v}|_2 \\
&\quad \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\rho\|_2\|\rho\|_3\|\mathbf{v}\|_0^2 \int_\Omega \rho\mathbf{f}_t\cdot\mathbf{v} \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon|\mathbf{f}_t|_{6/5}^2,
\end{aligned}$$

$$\int_\Omega \eta|\mathbf{v}|^2 \leqslant |\eta|_6|\mathbf{v}|_3|\mathbf{v}|_2 \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\eta\|_1^{4/3}\|\mathbf{v}\|_0^2,$$

$$\begin{aligned}
\int_\Omega \eta\mathbf{f}\cdot\mathbf{v} &\leqslant |\eta|_4|\mathbf{f}|_2|\mathbf{v}|_4 \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\eta\|_1^{4/3}\|\mathbf{f}\|_0^{4/3}\|\mathbf{v}\|_0^{2/3} \\
&\leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\mathbf{f}\|_0^{4/3}\big(\|\eta\|_1^2 + \|\mathbf{v}\|_0^2\big).
\end{aligned}$$

$$\lambda^2\int_\Omega \frac{1}{\rho^2}\eta\nabla\rho\otimes\nabla\rho : \nabla\mathbf{v} \leqslant C\|\eta\|_1\|\rho\|_2^2\|\mathbf{v}\|_1 \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\rho\|_2^4\|\eta\|_1^2,$$

$$\lambda^2\int_\Omega \frac{1}{\rho}(\nabla\eta\otimes\nabla\rho + \nabla\rho\otimes\nabla\eta) : \nabla\mathbf{v} \leqslant \varepsilon\|\mathbf{v}\|_1^2 + C_\varepsilon\|\rho\|_2\|\rho\|_3\|\eta\|_1^2.$$

Thus we obtain, choosing small enough ε and using estimates of Theorem 3.2, the following inequality holds:

$$a_n'(t) + b_n(t) \leqslant c_n(t)a_n(t) + d_n(t)a_{n-1}(t) + e_n(t), \tag{40}$$

where, we have defined

$$a_n(t) = \left\|\sqrt{\rho}\mathbf{v}\right\|_0^2, \qquad b_n(t) = \|\mathbf{v}\|_1^2, \qquad d_n(t) = C\|\mathbf{u}\|_2 \in L_t^2,$$
$$c_n(t) = C\big(\|\rho\|_3 + \|\eta\|_1^{4/3} + \|\mathbf{f}\|_0^{4/3}\big) \in L_t^1 \quad \big(\text{in fact } \in L_t^{3/2}\big),$$
$$e_n(t) = \varepsilon\|\eta\|_2^2 + C\big(1 + \|\mathbf{u}\|_2 + \|\rho\|_3 + \|\mathbf{f}\|_0^{4/3}\big)\|\eta\|_1^2 + C|\mathbf{f}_t|_{6/5}^2.$$

By hypothesis and estimates obtained above, we have $e_n(t)$ is bounded in L_t^1. Moreover, $a_0 = \|\sqrt{\rho^0}\mathbf{u}_t^0\|_0^2$ and $a_n(0) = \|\sqrt{\rho^n(0)}\mathbf{u}_t^n(0)\|_0^2 = \|\sqrt{\rho^0}\mathbf{u}_t^0\|_0^2$. Then, multiplying (5) evaluated at $t = 0$ by $\mathbf{u}_t^n(0)$,

$$\left\|\sqrt{\rho^0}\mathbf{u}_t^0\right\|_0^2 \leqslant C\big(\|\mathbf{u}_0\|_1^3\|\mathbf{u}_0\|_2 + \|\mathbf{u}_0\|_2^2 + \|\mathbf{u}_0\|_2^2\|\rho_0\|_2\|\rho_0\|_3 + \|\rho_0\|_2\|\rho_0\|_3^3 + \left\|\mathbf{f}(0)\right\|_0^2\big).$$

Therefore, hypotheses $\mathbf{u}_0 \in H^2$, $\rho_0 \in H^3$ and $\mathbf{f}(0) \in L^2$ imply $a_n(0) \leqslant A$. Thus we can apply the Gronwall's lemma with recurrence and one obtains (34). To find estimates in $L^\infty(H^2 \times H^1)$ for $(\mathbf{u}^n, q^n)$ we use the $H^2 \times H^1$ regularity of Stokes problem verified by $(\mathbf{u}^n, q^n)$, getting (as in (26))

$$\begin{aligned}\beta\big(\left\|\mathbf{u}^n\right\|_2^2 + \left\|q^n\right\|_1^2\big) &\leqslant C\big(\|\mathbf{f}\|_0^2 + \big(\left\|\mathbf{u}^{n-1}\right\|_1^4 + \left\|\rho^n\right\|_2^4\big)\left\|\mathbf{u}^n\right\|_1^2\big) \\ &\quad + C\big(\left\|\mathbf{u}_t^n\right\|_0^2 + \left\|\rho^n\right\|_2^6 + \left\|\rho^n\right\|_2^3\left\|\rho^n\right\|_3\big) \in L_t^\infty,\end{aligned}$$

hence $\mathbf{u}^n$ is bounded in $L^\infty(H^2)$ and q^n in $L^\infty(H^1)$.

We have that $\mathbf{u}_t^n$ is bounded in $L^2(H^1)$ and the rest of the second member $\mathbf{F}$ of Stokes problem verified by $(\mathbf{u}^n, q^n)$ is bounded in $L^2(H^1)$, therefore using $H^3 \times H^2$-regularity of Stokes problem, we deduce

$$\mathbf{u}^n \quad \text{is bounded in } L^2\big(H^3\big) \quad \text{and} \quad q^n \quad \text{in } L^2\big(H^2\big),$$

and the proof of (35) is completed. □

Finally, we will obtain scheme estimates with one order more of regularity than in Theorem 3.3. Velocity and pressure estimates will be only verified for strictly positive times.

Theorem 3.4. *Assume hypotheses of Theorem* 3.3. *If $\rho_0 \in H^4(\Omega)$, then the following estimates hold*:

$$\rho_t^n \quad \text{in } L^\infty\big(H^2\big) \cap L^2\big(H^3\big), \qquad \rho_{tt}^n \quad \text{in } L^2\big(H^1\big), \tag{41}$$
$$\rho^n \quad \text{in } L^\infty\big(H^4\big) \cap L^2\big(H^5\big). \tag{42}$$

Moreover, if $\mathbf{f}_t \in L^2(0,T;L^2)$ and $\mathbf{f} \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$, then the following estimates hold:

$$\sqrt{\sigma(t)}\mathbf{u}_t^n \quad \text{in } L^\infty\big(H^1\big) \cap L^2\big(H^2\big), \qquad \sqrt{\sigma(t)}\big(q_t^n, \mathbf{u}_{tt}^n\big) \quad \text{in } L^2\big(H^1 \times L^2\big), \tag{43}$$
$$\sqrt{\sigma(t)}\big(\mathbf{u}^n, q^n\big) \quad \text{in } L^\infty\big(H^3 \times H^2\big) \cap L^2\big(H^4 \times H^3\big) \tag{44}$$

(*recall that* $\sigma(t) = \min\{1, t\}$).

Proof. The main idea of the proof is the following: (41) is obtained doing strong estimates in the ρ_t^n-problem and (42) is deduced from (41) and regularity results for the Poisson problem associated to ρ^n. Afterwards, (43) are obtained doing strong estimates in the $(\mathbf{u}_t^n, q_t^n)$-problem and (44) is deduced from (43) and the Stokes problem associated to $(\mathbf{u}^n, q^n)$.

To prove (41) we need strong estimates on η (recall that $\eta = \rho_t^n$). We use the same arguments as in the proof of Lemma 3.1, but now for the η-problem $(31)_1$. Using the fact that

$$\int_\Omega \left|\nabla B(\bar{\mathbf{v}}, \nabla\rho)\right|^2 \leqslant C\left(\|\nabla\bar{\mathbf{v}}\cdot\nabla\rho\|_0^2 + \left\|(\bar{\mathbf{v}}\cdot\nabla)\nabla\rho\right\|_0^2\right) \leqslant C\|\bar{\mathbf{v}}\|_1^2\|\rho\|_2\|\rho\|_3,$$

$$\int_\Omega \left|\nabla B(\bar{\mathbf{u}}, \nabla\eta)\right|^2 \leqslant C\left(\|\nabla\bar{\mathbf{u}}\cdot\nabla\eta\|_0^2 + \left\|(\bar{\mathbf{u}}\cdot\nabla)\nabla\eta\right\|_0^2\right) \leqslant C\|\bar{\mathbf{u}}\|_1\|\bar{\mathbf{u}}\|_2\|\eta\|_2^2,$$

we get (using scheme estimates of Theorem 3.3)

$$\begin{aligned}\|\eta_t\|_1^2 + \frac{d}{dt}\|\eta\|_2^2 + \|\eta\|_3^2 &\leqslant C\left(\|\bar{\mathbf{u}}\|_1\|\bar{\mathbf{u}}\|_2\|\eta\|_2^2 + \|\rho\|_2\|\rho\|_3\|\bar{\mathbf{v}}\|_1^2\right)\\ &\leqslant C\left(\|\eta\|_2^2 + \|\bar{\mathbf{v}}\|_1^2\right).\end{aligned} \tag{45}$$

In order to bound $\|\eta(0)\|_2^2$, we take H^2-norm in (4) evaluated at $t=0$:

$$\left\|\eta(0)\right\|_2^2 = \left\|\rho_t^n(0)\right\|_2 \leqslant C\left(\|\mathbf{u}_0\|_2\|\rho_0\|_4 + \|\rho_0\|_4\right).$$

Therefore, Gronwall's lemma implies (41). Now, (42) can be easily deduced from H^4 and H^5 regularity of problem (37).

In order to obtain strong estimates for $(\mathbf{v}_t, q_t)$, one rewrites $(31)_2$ as

$$\rho\mathbf{v}_t - \mu\Delta\mathbf{v} + \nabla q_t = \mathbf{G}, \qquad \operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}|_\Sigma = 0, \qquad \mathbf{v}(0) = \mathbf{u}_t(0), \tag{46}$$

where

$$\begin{aligned}\mathbf{G} = &-\rho B(\bar{\mathbf{u}}, \nabla\mathbf{v}) - \rho B(\bar{\mathbf{v}}, \nabla\mathbf{u}) - \eta B(\bar{\mathbf{u}}, \nabla\mathbf{u})\\ &+ \lambda\left(C(\nabla\mathbf{u}, \nabla\eta) + C(\nabla\mathbf{v}, \nabla\rho)\right) - \eta\mathbf{v} + \eta\mathbf{f} + \rho\mathbf{f}_t\\ &+ \lambda^2\operatorname{div}\left[-\frac{1}{\rho^2}\eta\nabla\rho\otimes\nabla\rho + \frac{1}{\rho}(\nabla\eta\otimes\nabla\rho + \nabla\rho\otimes\nabla\eta)\right].\end{aligned}$$

Now, one multiplies (46) by $\mathbf{v}_t$ and one integrates on Ω,

$$m\|\mathbf{v}_t\|_0^2 + \frac{d}{dt}\|\nabla\mathbf{v}\|_0^2 \leqslant \|\mathbf{G}\|_0^2. \tag{47}$$

Bounding $\|\mathbf{G}\|_0^2$:

$$\|\rho\mathbf{f}_t\|_0^2 \leqslant C\|\mathbf{f}_t\|_0^2, \qquad \|\eta\mathbf{v}\|_0^2 \leqslant C|\mathbf{v}|_6^2\|\eta\|_2\|\eta\|_3, \qquad \|\eta\mathbf{f}\|_0^2 \leqslant C\|\eta\|_1\|\eta\|_2\|\mathbf{f}\|_0^2,$$

$$\left\|\rho B(\bar{\mathbf{u}}, \nabla\mathbf{v})\right\|_0^2 \leqslant C\|\nabla\mathbf{v}\|_0^2\|\bar{\mathbf{u}}\|_1\|\bar{\mathbf{u}}\|_2, \qquad \left\|\rho B(\bar{\mathbf{v}}, \nabla\mathbf{u})\right\|_0^2 \leqslant C\|\bar{\mathbf{v}}\|_1^2\|\mathbf{u}\|_1\|\mathbf{u}\|_2,$$

$$\left\|\eta B(\bar{\mathbf{u}}, \nabla\mathbf{u})\right\|_0^2 \leqslant C|\eta|_6^2|\bar{\mathbf{u}}|_\infty^2|\nabla\mathbf{u}|_3^2 \leqslant C\|\bar{\mathbf{u}}\|_1\|\bar{\mathbf{u}}\|_2\|\mathbf{u}\|_1\|\mathbf{u}\|_2\|\eta\|_1^2,$$

$$\left\|C(\nabla\mathbf{u}, \nabla\eta)\right\|_0^2 \leqslant C\|\mathbf{u}\|_1\|\mathbf{u}\|_2\|\eta\|_2^2, \qquad \left\|C(\nabla\mathbf{v}, \nabla\rho)\right\|_0^2 \leqslant C\|\mathbf{v}\|_1^2\|\rho\|_2\|\rho\|_3$$

$$\begin{aligned}&\left|\operatorname{div}\left[-\frac{1}{\rho^2}\eta\nabla\rho\otimes\nabla\rho + \frac{1}{\rho}(\nabla\eta\otimes\nabla\rho + \nabla\rho\otimes\nabla\eta)\right]\right|^2\\ &\quad\leqslant C\left(\|\rho\|_2^6\|\eta\|_1\|\eta\|_2 + \|\rho\|_2^4\|\eta\|_2^2 + \|\rho\|_2^3\|\rho\|_3\|\eta\|_1\|\eta\|_2 + \|\rho\|_2\|\rho\|_3\|\eta\|_2^2\right).\end{aligned}$$

Therefore, applying L^∞ estimates obtained above to previous bounds,

$$\|\mathbf{v}_t\|_0^2 + \frac{d}{dt}\|\mathbf{v}\|_1^2 \leqslant C\big(\|\mathbf{v}\|_1^2 + \|\mathbf{f}\|_0^2 + \|\eta\|_2^2 + \|\bar{\mathbf{v}}\|_1^2 + \|\mathbf{f}_t\|_0^2\big). \tag{48}$$

Applying H^2 regularity to the stationary problem related to (46),

$$\|\mathbf{v}\|_2^2 + \|q_t\|_1^2 \leqslant C\big(\|\mathbf{G}\|_0^2 + \|\mathbf{v}_t\|_0^2\big). \tag{49}$$

Thus, we obtain, by an adequate combination between (49) and (48) (eliminating $\|\mathbf{v}_t\|_0^2$ at the right-hand side),

$$\|\mathbf{v}_t\|_0^2 + \frac{d}{dt}\|\mathbf{v}\|_1^2 + \|\mathbf{v}\|_2^2 + \|q_t\|_1^2 \leqslant C\big(\|\mathbf{v}\|_1^2 + \|\mathbf{f}\|_0^2 + \|\eta\|_2^2 + \|\bar{\mathbf{v}}\|_1^2 + \|\mathbf{f}_t\|_0^2\big). \tag{50}$$

It is well known [6] that, there is no control about $\mathbf{v}(0) = \mathbf{u}_t(0)$ in the H^1-norm (only if initial data verify an overdeterminated global problem, which is not possible to verify in practice). Then, it will be necessary to consider only positive times, introducing for instance the cut-off function in $t = 0$, $\sigma(t) = \min\{1, t\}$. Then, multiplying (50) by $\sigma(t)$, (43) can be deduced from Lemma 2.2. Finally, (43) and the regularity of Stokes problem verified by $(\mathbf{u}^n, p^n)$, imply (44). $\square$

4. Error estimates

We use the notations $\mathbf{u}^{(n,s)} = \mathbf{u}^{n+s} - \mathbf{u}^n$, $q^{(n,s)} = q^{n+s} - q^n$ and $\rho^{(n,s)} = \rho^{n+s} - \rho^n$. Then, the problems satisfied by these differences are:

$$\rho_t^{(n,s)} - \lambda\Delta\rho^{(n,s)} = \big(\mathbf{u}^{(n-1,s)}\cdot\nabla\big)\rho^{n+s} + \big(\mathbf{u}^{n-1}\cdot\nabla\big)\rho^{(n,s)}, \tag{51}$$

$$\frac{\partial\rho^{(n,s)}}{\partial\mathbf{n}}\bigg|_{\Sigma_T} = 0, \qquad \rho^{(n,s)}\big|_{t=0} = 0, \tag{52}$$

and

$$\begin{aligned}
&\rho^n\mathbf{u}_t^{(n,s)} - \mu\Delta\mathbf{u}^{(n,s)} + \nabla q^{(n,s)} \\
&\quad = -\rho^{(n,s)}\mathbf{u}_t^{n+s} + \rho^{(n,s)}\mathbf{f} \\
&\qquad - \big(\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla\big)\mathbf{u}^{n+s} - \big(\rho^n\mathbf{u}^{(n-1,s)}\cdot\nabla\big)\mathbf{u}^{n+s} - \big(\rho^n\mathbf{u}^{n-1}\cdot\nabla\big)\mathbf{u}^{(n,s)} \\
&\qquad + \lambda\big(C\big(\nabla\mathbf{u}^{(n,s)}, \nabla\rho^n\big) + C\big(\nabla\mathbf{u}^{n+s}, \nabla\rho^{(n,s)}\big)\big) \\
&\qquad + \lambda^2\operatorname{div}\left[\left(\frac{1}{\rho}\right)^{(n,s)}\nabla\rho^{n+s}\otimes\nabla\rho^{n+s}\right] \\
&\qquad + \lambda^2\operatorname{div}\left[\frac{1}{\rho^n}\big(\nabla\rho^{(n,s)}\otimes\nabla\rho^{n+s} + \nabla\rho^n\otimes\nabla\rho^{(n,s)}\big)\right]
\end{aligned} \tag{53}$$

$$\operatorname{div}\mathbf{u}^{(n,s)} = 0, \tag{54}$$

$$\mathbf{u}^{(n,s)}\big|_{\Sigma_T} = 0, \qquad \mathbf{u}^{(n,s)}\big|_{t=0} = 0. \tag{55}$$

4.1. *Proof of Theorem 1.1*

It suffices to prove rate estimates (8) and (9) for $(\rho^{(n,s)}, \mathbf{u}^{(n,s)})$, i.e.,

$$\left\|\left(\rho^{(n,s)},\mathbf{u}^{(n,s)}\right)(t)\right\|_{H^1\times L^2}^2\leqslant G(n),\tag{56}$$

$$\int_0^t\left(\left\|\left(\rho^{(n,s)},\mathbf{u}^{(n,s)}\right)(\tau)\right\|_{H^2\times H^1}^2+\left\|\rho_t^{(n,s)}(\tau)\right\|_{L^2}^2\right)d\tau\leqslant G(n),\tag{57}$$

because in particular these estimates imply that $(\rho^n,\mathbf{u}^n)$ is a Cauchy sequence in $L^\infty(H^1\times L^2)\cap L^2(H^2\times H^1)$. Therefore, the whole sequences $(\rho^n,\mathbf{u}^n)\to(\rho,\mathbf{u})$ in $L^\infty(H^1\times L^2)\cap L^2(H^2\times H^1)$, hence with a limit argument in the iterative scheme (4)–(5) and taking into account estimates of Theorem 3.2, we can arrive at the unique strong solution $(\rho,\mathbf{u},p)$ of problem (1). Moreover, taking limit as $s\to+\infty$ in (56) and (57), since estimates (56)–(57) are uniform with respect to s, one has that the error $(\rho^n-\rho,\mathbf{u}^n-\mathbf{u})$ verifies rate estimates (8) and (9).

Consequently, we are going to prove (56) and (57).

Multiplying Eq. (51) by $-\lambda\Delta\rho^{(n,s)}$ and by $\rho_t^{(n,s)}$, we obtain (using L_t^∞ scheme estimates of Theorem 3.2)

$$\begin{aligned}
&\lambda\frac{d}{dt}\left|\nabla\rho^{(n,s)}\right|_2^2+\frac{\lambda^2}{2}\left|\Delta\rho^{(n,s)}\right|_2^2+\frac{1}{2}\left|\rho_t^{(n,s)}\right|_2^2\\
&\quad\leqslant C\left(\left|\left(\mathbf{u}^{(n-1,s)}\cdot\nabla\right)\rho^{n+s}\right|_2^2+\left|\left(\mathbf{u}^{n-1}\cdot\nabla\right)\rho^{(n,s)}\right|_2^2\right)\\
&\quad\leqslant C\left(\left\|\rho^{n+s}\right\|_2\left\|\rho^{n+s}\right\|_3\left\|\mathbf{u}^{(n-1,s)}\right\|_0^2+\left\|\mathbf{u}^{n-1}\right\|_1\left\|\mathbf{u}^{n-1}\right\|_2\left\|\rho^{(n,s)}\right\|_1^2\right)\\
&\quad\leqslant C\left(\left\|\rho^{n+s}\right\|_3\left\|\mathbf{u}^{(n-1,s)}\right\|_0^2+\left\|\mathbf{u}^{n-1}\right\|_2\left\|\rho^{(n,s)}\right\|_1^2\right).
\end{aligned}$$

Multiplying the velocity equation (53) by $\mathbf{u}^{(n,s)}$, integrating in Ω, and using the equality (which is deduced using Eq. (4)),

$$\left(\rho^n\mathbf{u}_t^{(n,s)}+\left(\rho^n\mathbf{u}^{n-1}\cdot\nabla\right)\mathbf{u}^{(n,s)}-\lambda\left(\nabla\rho^n\cdot\nabla\right)\mathbf{u}^{(n,s)},\mathbf{u}^{(n,s)}\right)=\frac{1}{2}\frac{d}{dt}\left|\sqrt{\rho^n}\mathbf{u}^{(n,s)}\right|_2^2$$

we have

$$\begin{aligned}
&\frac{d}{dt}\left|\sqrt{\rho^n}\mathbf{u}^{(n,s)}\right|_2^2+\mu\left|\nabla\mathbf{u}^{(n,s)}\right|_2^2\\
&\quad\leqslant C\left|\rho^{(n,s)}\right|_3^2\left(\left|\mathbf{u}_t^{n+s}\right|_2^2+|\mathbf{f}|_2^2\right)\\
&\qquad+C\left(\left|\rho^{(n,s)}\right|_6^2\left|\mathbf{u}^{n-1+s}\right|_6^2\left|\nabla\mathbf{u}^{n+s}\right|_2^2+\left|\rho^n\right|_\infty^2\left|\mathbf{u}^{(n-1,s)}\right|_2^2\left|\nabla\mathbf{u}^{n+s}\right|_3^2\right)\\
&\qquad+C\lambda^2\left(\left|C\left(\nabla\mathbf{u}^{(n,s)},\nabla\rho^n\right)\right|_{6/5}^2+\left|C\left(\nabla\mathbf{u}^{n+s},\nabla\rho^{(n,s)}\right)\right|_{6/5}^2\right)\\
&\qquad+C\lambda^4\left(\left|\rho^{(n,s)}\nabla\rho^{n+s}\otimes\nabla\rho^{n+s}\right|_2^2+\left|\nabla\rho^{(n,s)}\right|_2^2\left|\nabla\rho^{n+s}\right|_\infty^2\right).
\end{aligned}$$

Then, using L_t^∞ scheme estimates of Theorem 3.2,

$$\begin{aligned}
&\frac{d}{dt}\left|\sqrt{\rho^n}\mathbf{u}^{(n,s)}\right|_2^2+\mu\left|\nabla\mathbf{u}^{(n,s)}\right|_2^2\\
&\quad\leqslant C\left(\left\|\mathbf{u}_t^{n+s}\right\|_0^2+\|\mathbf{f}\|_0^2\right)\left\|\rho^{(n,s)}\right\|_1^2\\
&\qquad+C\left(\left\|\rho^{(n,s)}\right\|_1^2+\left\|\mathbf{u}^{n+s}\right\|_1\left\|\mathbf{u}^{n+s}\right\|_2\left\|\mathbf{u}^{(n-1,s)}\right\|_0^2\right)\\
&\qquad+C\lambda^2\left(\left\|\rho^n\right\|_2\left\|\rho^n\right\|_3\left\|\mathbf{u}^{(n,s)}\right\|_1^2+\left\|\mathbf{u}^{n+s}\right\|_1\left\|\mathbf{u}^{n+s}\right\|_2\left\|\rho^{(n,s)}\right\|_1^2\right)\\
&\qquad+C\lambda^4\left(\left\|\rho^{n+s}\right\|_2^2+\left\|\rho^{n+s}\right\|_2\left\|\rho^{n+s}\right\|_3\right)\left\|\rho^{(n,s)}\right\|_1^2
\end{aligned}$$

$$\begin{aligned}&\leqslant C\big(\big\|\mathbf{u}_t^{n+s}\big\|_0^2+\|\mathbf{f}\|_0^2+1+\big\|\mathbf{u}^{n+s}\big\|_2+\big\|\rho^{n+s}\big\|_3\big)\big\|\rho^{(n,s)}\big\|_1^2\\&\quad+C\big\|\mathbf{u}^{n+s}\big\|_2\big\|\mathbf{u}^{(n-1,s)}\big\|_0^2+C\big\|\rho^n\big\|_3\big\|\mathbf{u}^{(n,s)}\big\|_1^2.\end{aligned}$$

Adding the previous inequalities, we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\big|\sqrt{\rho^n}\mathbf{u}^{(n,s)}\big|_2^2+\lambda\big|\nabla\rho^{(n,s)}\big|_2^2\big)+\mu\big|\nabla\mathbf{u}^{(n,s)}\big|_2^2+\frac{\lambda^2}{2}\big|\Delta\rho^{(n,s)}\big|_2^2+\frac{1}{2}\big|\rho_t^{(n,s)}\big|_2^2\\&\quad\leqslant\psi_{n,s}(t)\big|\mathbf{u}^{(n-1,s)}\big|_2^2+\varphi_{n,s}(t)\big(\big|\sqrt{\rho^n}\mathbf{u}^{(n,s)}\big|_2^2+\big|\nabla\rho^{(n,s)}\big|_2^2\big),\end{aligned}$$

where

$$\begin{aligned}&\psi_{n,s}(t)=C\big(\big\|\rho^{n+s}\big\|_3+\big\|\mathbf{u}^{n+s}\big\|_2\big),\\&\varphi_{n,s}(t)=C\big(\big\|\mathbf{u}^{n-1}\big\|_2+\big\|\mathbf{u}_t^{n+s}\big\|_0^2+\|\mathbf{f}\|_0^2+1+\big\|\rho^{n+s}\big\|_3+\big\|\mathbf{u}^{n+s}\big\|_2+\big\|\rho^n\big\|_3\big).\end{aligned}$$

From L_t^2 estimates of $(\rho^n,\mathbf{u}^n)$ given in Theorem 3.2 (see (29)–(30)), $(\psi_{n,s})$ is bounded in $L^2(0,T)$ and $(\varphi_{n,s})$ is bounded in $L^1(0,T)$ (since $\|\mathbf{f}\|_0^2\in L^1(0,T)$). Therefore, Lemma 2.2 implies (recalling that $|\mathbf{u}^{(n,s)}(0)|=0$ and $|\nabla\rho^{(n,s)}(0)|=0$ and applying again estimates (29)–(30) given in Theorem 3.2)

$$\begin{aligned}&\big(\big|\sqrt{\rho^n}\mathbf{u}^{(n,s)}\big|_2^2+\lambda\big|\nabla\rho^{(n,s)}\big|_2^2\big)(t)+\int_0^t\Big(\mu\big|\nabla\mathbf{u}^{(n,s)}\big|_2^2+\frac{\lambda^2}{2}\big|\Delta\rho^{(n,s)}\big|_2^2+\frac{1}{2}\big|\rho_t^{(n,s)}\big|_2^2\Big)\\&\quad\leqslant E\Big\|\big|\sqrt{\rho^0}\mathbf{u}^{(0,s)}\big|_2^2+\lambda\big|\nabla\rho^{(0,s)}\big|_2^2\Big\|_{L^\infty(0,t)}\bigg[\frac{(Dt)^n}{n!}\bigg]^{1/2}\leqslant C\bigg[\frac{(Dt)^n}{n!}\bigg]^{1/2},\end{aligned}$$

hence the estimates (56)–(57) hold, and the proof is finished.

4.2. *Proof of Theorem 1.2*

Again, it suffices to prove (10)–(11) changing the error $(\rho^n-\rho,\mathbf{u}^n-\mathbf{u},q^n-q)$ by $(\rho^{(n,s)},\mathbf{u}^{(n,s)},q^{(n,s)})$. Multiplying the density error equation (51) by $-\lambda\Delta\rho_t^{(n,s)}$ and taking gradient of (51) multiplied by $\nabla\Delta\rho^{(n,s)}$ (arguing as in Lemma 3.1),

$$\begin{aligned}&\lambda\frac{d}{dt}\big|\Delta\rho^{(n,s)}\big|_2^2+\frac{\lambda^2}{2}\big|\nabla\Delta\rho^{(n,s)}\big|_2^2+\frac{1}{2}\big|\nabla\rho_t^{(n,s)}\big|_2^2\\&\quad\leqslant C\Big(\big|\nabla\big(\mathbf{u}^{(n-1,s)}\cdot\nabla\rho^{n+s}\big)\big|^2+\big|\nabla\big(\mathbf{u}^{n-1}\cdot\nabla\rho^{(n,s)}\big)\big|^2\Big)\\&\quad\leqslant C\Big(\big\|\rho^{n+s}\big\|_2\big\|\rho^{n+s}\big\|_3\big\|\mathbf{u}^{(n-1,s)}\big\|_1^2+\big\|\mathbf{u}^{n-1}\big\|_1\big\|\mathbf{u}^{n-1}\big\|_2\big\|\rho^{(n,s)}\big\|_2^2\Big)\\&\quad\leqslant C\Big(\big\|\rho^{n+s}\big\|_3\big\|\mathbf{u}^{(n-1,s)}\big\|_1^2+\big\|\mathbf{u}^{n-1}\big\|_2\big\|\rho^{(n,s)}\big\|_2^2\Big).\end{aligned}$$

Multiplying the velocity equation (53) by $\mathbf{u}_t^{(n,s)}$ and balancing with the $H^2\times H^1$ regularity of Stokes problem satisfied by $(\mathbf{u}^{(n,s)},p^{(n,s)})$ (arguing again as in Lemma 3.1), we have

$$\begin{aligned}&\frac{m}{2}\big|\mathbf{u}_t^{(n,s)}\big|_2^2+\mu\frac{d}{dt}\big|\nabla\mathbf{u}^{(n,s)}\big|_2^2+\beta\big(\big|\Delta\mathbf{u}^{(n,s)}\big|_2^2+\big|\nabla q^{(n,s)}\big|_2^2\big)\\&\quad\leqslant C\Big(\big|\rho^{(n,s)}\mathbf{u}_t^{n+s}\big|_2^2+\big|\big(\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla\big)\mathbf{u}^{n+s}\big|_2^2+\big|\big(\rho^n\mathbf{u}^{(n-1,s)}\cdot\nabla\big)\mathbf{u}^{n+s}\big|_2^2\Big)\end{aligned}$$

$$+ C\big|\big(\rho^n\mathbf{u}^{n-1}\cdot\nabla\big)\mathbf{u}^{(n,s)}\big|_2^2 + C\lambda^2\Big(\big|C\big(\nabla\mathbf{u}^{n+s},\nabla\rho^{(n,s)}\big)\big|_2^2 + \big|C\big(\nabla\mathbf{u}^{(n,s)},\nabla\rho^n\big)\big|_2^2\Big)$$
$$+ C\lambda^4\left|\operatorname{div}\left[\left(\frac{1}{\rho}\right)^{(n,s)}\nabla\rho^{n+s}\otimes\nabla\rho^{n+s}\right]\right|_2^2 + C\big|\rho^{(n,s)}\mathbf{f}\big|_2^2$$
$$+ C\lambda^4\left|\operatorname{div}\left[\frac{1}{\rho^n}\big(\nabla\rho^{(n,s)}\otimes\nabla\rho^{n+s} + \nabla\rho^n\otimes\nabla\rho^{(n,s)}\big)\right]\right|_2^2.$$

Estimating the right-hand side of the above inequality (using $L^\infty(0,T)$ estimates for $(\rho^n,\mathbf{u}^n)$ given in Theorem 3.2) and adding with the inequality for the density, we have

$$\begin{aligned}
&\frac{d}{dt}\big(\mu\big|\nabla\mathbf{u}^{(n,s)}\big|_2^2 + \lambda\big|\Delta\rho^{(n,s)}\big|_2^2\big) + \frac{1}{2}\big|\nabla\rho_t^{(n,s)}\big|_2^2 + \frac{\lambda^2}{2}\big|\nabla\Delta\rho^{(n,s)}\big|_2^2 \\
&\quad + \frac{m}{2}\big|\mathbf{u}_t^{(n,s)}\big|_2^2 + \beta\big(\big|\Delta\mathbf{u}^{(n,s)}\big|_2^2 + \big|\nabla q^{(n,s)}\big|_2^2\big) \\
&\quad \leqslant \eta_1(t)\big|\nabla\mathbf{u}^{(n-1,s)}\big|_2^2 + \eta_2(t)\big|\Delta\rho^{(n,s)}\big|_2^2 + \eta_3(t)\big|\nabla\mathbf{u}^{(n,s)}\big|_2^2,
\end{aligned}\tag{58}$$

where

$$\begin{aligned}
\eta_1(t) &= C\big(\big\|\mathbf{u}^{n+s}(t)\big\|_2 + \big\|\rho^{n+s}(t)\big\|_3\big),\\
\eta_2(t) &= C\big(\big\|\mathbf{u}_t^{n+s}(t)\big\|_0^2 + \big\|\mathbf{u}^{n-1+s}(t)\big\|_2 \\
&\quad + \big\|\mathbf{u}^{n+s}(t)\big\|_2 + \big\|\mathbf{f}(t)\big\|_0^2 + \big\|\mathbf{u}^{n-1}(t)\big\|_2 + 1 + \big\|\rho^n(t)\big\|_3 + \big\|\rho^{n+s}(t)\big\|_3\big),\\
\eta_3(t) &= C\big(\big\|\mathbf{u}^{n-1}\big\|_2 + \big\|\rho^n(t)\big\|_3 + \big\|\rho^{n+s}(t)\big\|_3\big).
\end{aligned}$$

From estimates of Theorem 3.2 (see (29)–(30)), sequences η_1 and η_3 are bounded in $L^2(0,T)$ and η_2 is bounded in $L^1(0,T)$. Therefore, applying Lemma 2.2 we obtain the rates estimates (10)–(11) for $(\rho^{(n,s)},\mathbf{u}^{(n,s)},q^{(n,s)})$.

4.3. *Proof of Theorem 1.3*

Once more, it suffices to prove (12)–(13) for $\rho^{(n,s)}$. Differentiating the density error equation (51) with respect to t, multiplying by $-\lambda\Delta\rho_t^{(n,s)}$, we have (using estimates of Theorem 3.3)

$$\begin{aligned}
&\lambda\frac{d}{dt}\big|\nabla\rho_t^{(n,s)}\big|^2 + \lambda^2\big|\Delta\rho_t^{(n,s)}\big|^2 \\
&\quad\leqslant C\big(\big|\big(\mathbf{u}^{(n-1,s)}\cdot\nabla\rho^{n+s}\big)_t\big|_2^2 + \big|\big(\mathbf{u}^{n-1}\cdot\nabla\rho^{(n,s)}\big)_t\big|_2^2\big) \\
&\quad\leqslant C\big(\big\|\rho^{n+s}\big\|_2\big\|\rho^{n+s}\big\|_3\big\|\mathbf{u}_t^{(n-1,s)}\big\|_0^2 + \big\|\mathbf{u}^{(n-1,s)}\big\|_1\big\|\mathbf{u}^{(n-1,s)}\big\|_2\big\|\rho_t^{n+s}\big\|_1^2 \\
&\qquad + \big\|\mathbf{u}_t^{n-1}\big\|_0\big\|\mathbf{u}_t^{n-1}\big\|_1\big\|\rho^{(n,s)}\big\|_2^2 + \big\|\mathbf{u}^{n-1}\big\|_1\big\|\mathbf{u}^{n-1}\big\|_2\big\|\rho_t^{(n,s)}\big\|_1^2\big) \\
&\quad\leqslant C\big(\big\|\mathbf{u}_t^{(n-1,s)}\big\|_0^2 + \big\|\mathbf{u}^{(n-1,s)}\big\|_1\big\|\mathbf{u}^{(n-1,s)}\big\|_2 + \big\|\mathbf{u}_t^{n-1}\big\|_1\big\|\rho^{(n,s)}\big\|_2^2 + \big\|\rho_t^{(n,s)}\big\|_1^2\big).
\end{aligned}$$

Therefore, using estimates of Theorem 3.3 and error estimates of Theorem 1.2, the Gronwall's lemma implies (12). Finally, (13) is deduced from the H^3 and H^4 regularity of Poisson–Neumann problem satisfied by $\rho^{(n,s)}$.

4.4. Proof of Theorem 1.4

It suffices to prove (14)–(16) for $(\mathbf{u}^{(n,s)}, q^{(n,s)})$. Differentiating the velocity error equation (53) with respect to t, taking $\mathbf{u}_t^{(n,s)}$ as test function and using the equality

$$\big(\rho^n \mathbf{u}_{tt}^{(n,s)} + \big(\rho^n \mathbf{u}^{n-1}\cdot\nabla\big)\mathbf{u}_t^{(n,s)} - \lambda\big(\nabla\rho^n\cdot\nabla\big)\mathbf{u}_t^{(n,s)}, \mathbf{u}_t^{(n,s)}\big) = \frac{1}{2}\frac{d}{dt}\big|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}\big|_2^2,$$

we have

$$\begin{aligned}
&\frac{d}{dt}\big|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}\big|_2^2 + \mu\big|\nabla\mathbf{u}_t^{(n,s)}\big|_2^2\\
&\quad\leqslant \big|\rho_t^n\big|_3^2\big|\mathbf{u}_t^{(n,s)}\big|_2^2 + \big|\rho_t^{(n,s)}\big|_6^2\big|\mathbf{u}_t^{n+s}\big|_{3/2}^2 + \big|\rho^{(n,s)}\big|_\infty^2\big|\mathbf{u}_{tt}^{n+s}\big|_{6/5}^2\\
&\qquad + \big|\big(\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla\mathbf{u}^{n+s}\big)_t\big|_{6/5}^2 + \big|\big(\rho^n\mathbf{u}^{(n-1,s)}\cdot\nabla\mathbf{u}^{n+s}\big)_t\big|_{6/5}^2\\
&\qquad + \big|\big(\rho^n\mathbf{u}^{n-1}\big)_t\cdot\nabla\mathbf{u}^{(n,s)}\big|_{6/5}^2 + \lambda^2\big(\big|C\big(\nabla\mathbf{u}^{(n,s)},\nabla\rho^n\big)_t\big|_{6/5}^2 + \big|C\big(\nabla\mathbf{u}^{n+s},\nabla\rho^{(n,s)}\big)_t\big|_{6/5}^2\big)\\
&\qquad + \lambda^4\left|\left(\left(\frac{1}{\rho}\right)^{(n,s)}\nabla\rho^{n+s}\otimes\nabla\rho^{n+s} + \frac{1}{\rho^n}\big(\nabla\rho^{(n,s)}\otimes\nabla\rho^{n+s} + \nabla\rho^n\otimes\nabla\rho^{(n,s)}\big)\right)_t\right|_2^2\\
&\qquad + \big|\rho_t^{(n,s)}\big|_6^2|\mathbf{f}|_{3/2}^2 + \big|\rho^{(n,s)}\big|_\infty^2|\mathbf{f}_t|_{6/5}^2.
\end{aligned}$$

Estimating in a similar manner as in Theorem 3.4 and using estimates of Theorem 3.4, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\big|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}\big|_2^2 + \mu\big|\nabla\mathbf{u}_t^{(n,s)}\big|_2^2\\
&\quad\leqslant C\big|\nabla\rho_t^{(n,s)}\big|_2^2\big(\|\mathbf{f}\|_0^2+1\big) + C\big|\mathbf{u}_t^{(n,s)}\big|_2^2 + C\big|\mathbf{u}_t^{(n-1,s)}\big|_2^2\\
&\qquad + C\big|\Delta\rho^{(n,s)}\big|_2^2\big(\big\|\mathbf{u}_{tt}^{n+s}\big\|_0^2 + \big\|\mathbf{u}_t^n\big\|_1^2 + \big\|\mathbf{u}_t^{n+s}\big\|_1^2 + |\mathbf{f}_t|_{6/5}^2 + 1\big)\\
&\qquad + C\big|\nabla\mathbf{u}^{(n,s)}\big|_2^2\big(\big\|\rho_t^n\big\|_2^2 + \big\|\mathbf{u}_t^{n-1}\big\|_1\big) + C\big|\Delta\mathbf{u}^{(n-1,s)}\big|_2\big|\nabla\mathbf{u}^{(n-1,s)}\big|_2.
\end{aligned}$$

Notice that estimates will be only for positive times, because of the term $\|\mathbf{u}_{tt}^{n+s}\|_0^2$, which appears from the nonlinear term $\rho\mathbf{u}_t$. Therefore, the cut-off function $\sigma(t)$ must be introduced. Multiplying by $\sigma(t)=\min\{1,t\}$, recalling that $m|\mathbf{u}_t^{(n,s)}|^2 \leqslant |\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2 \leqslant M|\mathbf{u}_t^{(n,s)}|^2$ and $\sigma'(t)\leqslant 1$, we get

$$\begin{aligned}
&\frac{d}{dt}\big[\sigma(t)\big|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}\big|_2^2\big] + \sigma(t)\mu\big|\nabla\mathbf{u}_t^{(n,s)}\big|_2^2\\
&\quad\leqslant C\big[\sigma(t)\big|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}\big|^2\big] + \sigma'(t)\big[\big|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}\big|^2\big] + \big(1+\|\mathbf{f}\|_0^2\big) + b_n(t) + c_n(t),
\end{aligned}$$

where $\|b_n\|_{L^1(0,t)} \leqslant G(n)$ and $\|c_n\|_{L^1(0,t)} \leqslant G(n-1)$ thanks to Theorems 1.2, 1.3, 3.2–3.4. Then, using Gronwall's lemma, taking into account that $\sigma(0)=0$, we obtain (14).

Now, from the $H^2\times H^1$ regularity of Stokes problem satisfied by $(\mathbf{u}^{(n,s)}, p^{(n,s)})$ (see (53)), we obtain (bounding as in proof of Theorem 1.2)

$$\begin{aligned}
&\sigma(t)\big(\big\|\mathbf{u}^{(n,s)}(t)\big\|_2^2 + \big\|q^{(n,s)}(t)\big\|_1^2\big)\\
&\quad\leqslant C\sigma(t)\big(\big\|\mathbf{u}_t^{(n,s)}(t)\big\|_0^2\\
&\qquad + \eta_1(t)\big|\nabla\mathbf{u}^{(n-1,s)}(t)\big|_2^2 + \eta_2(t)\big|\Delta\rho^{(n,s)}(t)\big|_2^2 + \eta_3(t)\big|\nabla\mathbf{u}^{(n,s)}(t)\big|_2^2\big).
\end{aligned}$$

Thus, by using estimates (10), (12) and (14) we obtain (15) for $(\mathbf{u}^{(n,s)}, q^{(n,s)})$.

Finally, estimates (16) can be proved with analogous arguments, using now the $H^3\times H^2$ regularity of Stokes problem satisfied by $(\mathbf{u}^{(n,s)}, q^{(n,s)})$.

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